Problem Set 1

Quantum Field Theory

Problem 1. The Lie group of distance-preserving transformations of n-dimensional Euclidean space $\mathbb{R}^n$ is given by the orthogonal group $O(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid AA^T = I\}$. In $\mathbb{R}^3$, it can also be thought of as the group of reflections and rotations under composition. Determine what type of matrices are the generators of the corresponding Lie algebra $\mathfrak{o}(n)$.[1]

Problem 2. In Pauli's formalism for angular momentum, the spin operator is defined as $\vec{S} = \frac{\vec{\sigma}}{2}$, where $\vec{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices, obeying the commutation relations,

$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ij}{}_k \sigma^k$$

where the indices of ε are raised and lowered by δ^{ij} and δ_{ij} respectively.

- Using only the commutation relations of the Pauli matrices, find the Casimir operator of this spin-$\frac{1}{2}$ representation.
- Denoting by $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\theta, \cos\theta)$ a unit-norm vector, find the two-component eigen-spinor $|\psi_{\vec{n}}\rangle$ of the operator $\vec{S}\cdot\vec{n}$, with the eigenvalue $\frac{1}{2}$. There is no need to normalize the eigenstate $|\psi_{\vec{n}}\rangle$.

Note that Casimir operators of a Lie algebra are polynomials in the generators of a Lie algebra that commute with all the elements of the Lie algebra.

Problem 3. The six generators $J^{\mu\nu}$ of the Lie algebra of the Lorentz group $SO(3,1)$ in the four-vector representation are given by

$$(J^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)$$

where $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$. $(J^{\mu\nu})^\rho{}_\sigma$ denotes the ρ, σ entry of the $J^{\mu\nu}$ matrix. Notice that $J^{\mu\nu} = -J^{\nu\mu}$ and $(J^{\mu\nu})_{\rho\sigma} = -(J^{\mu\nu})_{\sigma\rho}$, where the index is lowered with the Minkowski space metric η.

1. Write explicitly the six generators $J^{\mu\nu}$ as 4×4 matrices.
2. Compute the commutators of two Lorentz generators $[J^{\mu\nu}, J^{\alpha\beta}]$.
3. Recall the three rotation generators are defined as $J^i = \frac{1}{2}\varepsilon^i{}_{jk}J^{jk}$ and the three boost generators are defined as $K^i = J^{i0}$. Compute the commutator $[K^i, K^j]$.

[1]Louis Strehlow literally begged me to change the wording of this question, even provided me with the new wording.

Problem 4. For any $n \times n$ matrices A, B, C, prove the identity (Jacobi identity):

$$[[A,B],C] = [A,[B,C]] + [B,[A,C]] = 0$$

Consider the Lie algebra with the structure constants $f^{ab}_{\;\;c}$, i.e. there is a basis T^a in the Lie algebra such that $[T^a, T^b] = if^{ab}_{\;\;c}T^c$.
The adjoint representation of the Lie algebra is defined by

$$(T^a_{adj})^b_{\;\;c} = -if^{ab}_{\;\;c}.$$

Using the Jacobi identity, verify that,

$$[T^a_{adj}, T^b_{adj}] = if^{ab}_{\;\;c}T^c_{adj}.$$

Problem 5. Write the commutation relations of the Poincaré algebra in terms of the operators $H = P^0, P^i, J^i, K^i$. You can use the representation of your choice.
Suppose that there is a dynamical system with Hamiltonian $\mathcal{H}$ and symmetries generated by the rest of the generators of the Poincaré algebra. From the algebra, identify the generators associated with conserved charges.

Problem Set 2

Quantum Field Theory

Problem 1. Two different bases in a Clifford algebra are related by a similarity transformation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}$$

and

$$\gamma^{\mu'} \gamma^{\nu'} + \gamma^{\nu'} \gamma^{\mu'} = 2\eta^{\mu'\nu'}\mathbf{1}$$

there exists S such that

$$\gamma^{\mu'} = \mathrm{S}\,\gamma^\mu\,\mathrm{S}^{-1}$$

Use this to show that

1. $\bar{\psi}\psi$ transforms as a scalar under Lorentz transformations
2. $\bar{\psi}\gamma^\mu\psi$ transforms as a vector under Lorentz transformations
3. $\bar{\psi}\gamma^{\mu\nu}\psi$ transforms as a tensor under Lorentz transformations where $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$
4. Find also the transformation of $\bar{\psi}\gamma_5\psi$, $\bar{\psi}\gamma_5\gamma^\mu\psi$ under both Lorentz and parity transformations.

where ψ is a Dirac spinor and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Problem 2. For a Lorentz invariant theory, the conserved current associated to the Lorentz transformations is

$$J^{(\rho\sigma)\mu} = x^\rho\Theta^{\mu\sigma} - x^\sigma\Theta^{\mu\rho}$$

where $\Theta^{\mu\nu}$ is the energy-momentum tensor of the theory.

In particular for the complex scalar field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$$

where ϕ^* is the complex conjugate of ϕ.

The energy-momentum tensor is

$$\Theta_{\mu\nu} = \partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi - \eta_{\mu\nu}\mathcal{L}$$

(i) Compute the conserved charges associated to spatial rotations

$$M^{ij} = \int d^3x \; J^{(ij)0}$$

Bring them into the form

$$M^{ij} = i \int d^3x \; [\phi^* L^{ij} \partial_0 \phi - \partial_0 \phi^* L^{ij} \phi]$$

for sufficiently decaying fields at infinity where

$$L^{ij} = i(x^i \partial^j - x^j \partial^i)$$

(ii) Define the bilinear form on the space of fields

$$\langle \phi_1 | \phi_2 \rangle = i \int d^3x \; (\phi_1^* \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1^*) \equiv i \int d^3x \; \phi_1^* \overleftrightarrow{\partial_0} \phi_2$$

Show that $\langle \phi_1 | \phi_2 \rangle$ is time-independent provided that ϕ_1, ϕ_2 satisfy the Klein-Gordon equation.

(iii) Demonstrate that

$$M^{ij} = \langle \phi \big| L^{ij} \phi \rangle$$

Also show that

$$P^\mu = \int d^3x \; \Theta^{0\mu} = \langle \phi | i \partial^\mu \phi \rangle$$

Problem 3. Let ψ be a Dirac spinor and consider the two different Lagrangians

$$\mathcal{L}_1 = \bar{\psi}(i \not{\partial} - m)\psi$$

$$\mathcal{L}_2 = \bar{\psi} \left(\frac{i}{2} \overleftrightarrow{\not{\partial}} \psi - m \right) \psi$$

(i) Compute the field equations associated with both Lagrangians.

(ii) Find the energy-momentum tensors associated with both $\mathcal{L}_1$ and $\mathcal{L}_2$.

(iii) Compute the difference in energy of the field configurations defined by the two different energy-momentum tensors i.e. compute $\Delta E = E_1 - E_2$.

Problem Set 3

Quantum Field Theory

Problem 1. The equation that describes the minimally coupled Dirac field ψ to the electromagnetic potential A_μ is

$$(i\partial\!\!\!/ - e\gamma^\mu A_\mu)\psi - m\psi = 0 \tag{1}$$

where e is the electric charge and γ^μ are the matrices

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_{2\times 2} & 0 \\ 0 & -\mathbf{1}_{2\times 2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

The non-relativistic limit of this equation is

$$i\frac{\partial}{\partial t}\phi = \left(\frac{1}{2m}(i\vec{\partial} - e\vec{A})^2 + eA_0 - \frac{e}{2m}\vec{\sigma}\cdot\vec{B}\right)\phi \tag{2}$$

where $\vec{B}$ is the magnetic field. Derive (2) from (1) using the approximations

$$i\partial_t\chi << m\chi \quad \text{and} \quad eA_0 << m$$

where $\psi = e^{-imt}\begin{pmatrix}\phi \\ \chi\end{pmatrix}$.

Problem 2. Consider $U^r(p)$ and $V^s(p)$ as in the notes. Verify the following:

(i) $\overline{U}^s(p)\,U^r(p) = 2m\delta^{rs}$

(ii) $\overline{V}^s(p)\,V^r(p) = -2m\delta^{rs}$

(iii) $\sum_{s=1,2} U^s(p)\,\overline{U}^s(p) = p\!\!\!/ + m\mathbf{1}_{4\times 4}$

(iv) $\sum_{s=1,2} V^s(p)\,\overline{V}^s(p) = p\!\!\!/ - m\mathbf{1}_{4\times 4}$

Problem 3. The Lagrangian of the electro-magnetic field is

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(i) Find the energy-momentum tensor $\Theta^{\mu\nu}$ of the theory.

(ii) Demonstrate the conservation of $\Theta^{\mu\nu}$ subject to the field equations.

(iii) Demonstrate that $\Theta^{\mu\nu}$ can be improved to $\tilde{\Theta}^{\mu\nu}$ such that $\tilde{\Theta}^{\mu\nu} = \tilde{\Theta}^{\nu\mu}$ and $\partial_\mu\tilde{\Theta}^{\mu\nu} = 0$.

Problem 4. The free real scalar field ϕ expressed in terms of the creation and annihilation operators can be written as

$$\phi(x) = \int \frac{đ^3 k}{2E_k} \left(a(k)e^{-ikx} + a^\dagger(k)e^{ikx} \right).$$

Express $a(k)$ and $a^\dagger(k)$ in terms of ϕ and its canonical momentum π.

Problem Set 4

Quantum Field Theory

Problem 1. The Dirac Hamiltonian for an electron in a central potential V is

$$\mathrm{H} = \vec{\alpha} \cdot \vec{\mathrm{p}} + \beta\, m + V(r)\, 1$$

where $r = |\vec{x}|$, $\alpha^k = \gamma^0 \gamma^k$ and $\beta = \gamma^0$

(i) Show that the operator H does not commute with the orbital angular momentum operator $\vec{\mathrm{L}} = \vec{\mathrm{x}} \times \vec{\mathrm{p}}$

(ii) Show that it does commute with $\vec{\mathrm{J}} = \vec{\mathrm{L}} + \frac{1}{2}\vec{\Sigma}$ where $\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $\vec{\Sigma}^k = \frac{1}{2}\epsilon^{kij}\Sigma^{ij}$

As usual $[\mathrm{x}^\mathrm{i}, \mathrm{p}^\mathrm{j}] = i\delta^{ij}\, 1$ (The canonical commutator relations).

Problem 2. For any 3 operators or matrices X, Y, Z show that

(1) $[[\mathrm{X}, \mathrm{Y}], \mathrm{Z}] = [\mathrm{X}, [\mathrm{Y}, \mathrm{Z}]] + [[\mathrm{X}, \mathrm{Z}], \mathrm{Y}] = \{\mathrm{X}, \{\mathrm{Y}, \mathrm{Z}\}\}$ - $\{\{\mathrm{X}, \mathrm{Z}\}, \mathrm{Y}\}$ where $[\mathrm{X}, \mathrm{Y}] =$ XY-YX and $\{\mathrm{X}, \mathrm{Y}\} = \mathrm{XY} + \mathrm{YX}$

(2) If $\mathrm{S}^{\mu\nu} = \frac{1}{2}\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ evaluate $[\mathrm{S}^{\mu\nu}, \gamma^\rho]$

(3) Evaluate $[\mathrm{S}^{\mu\nu}, \mathrm{S}^{\rho\sigma}]$ what do you observe?

(4) Express the matrices $\vec{\Sigma}^k = \frac{1}{2}\epsilon^{kij}\Sigma^{ij}$ in the standard basis.

Problem 3. Show that in the limit $m \to 0$, the solution u of the Dirac equation can be chosen to be an eigenvector of γ_5. In the standard representation find the appropriate linear combination of u^1 and u^2.

Show that they are also eigenvectors of the helicity operator

$$\mathrm{h} = \frac{\vec{\Sigma} \cdot \vec{\mathrm{p}}}{|\vec{\mathrm{p}}|}$$

Problem 4. Let ϕ be a quantised free scalar field.

(i) Show that

$$T^{\mu\nu} = :\partial^\mu\phi\, \partial^\nu\phi - \eta^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\phi\, \partial^\lambda\phi - \frac{1}{2}m^2\phi^2\right):$$

satisfies $\partial_\mu T^{\mu\nu} = 0$

(ii) Express

$$P^\nu = \int d^3x \, T^{0\nu}$$

in terms of the creation and the annihilation operators $a^*(k)$ and $a(k)$.

Problem Set 5

Quantum Field Theory

Problem 1. A one-particle state is given by

$$|\psi\rangle = \int \frac{đ^3k}{2E_{\mathbf{k}}} |\mathbf{k}\rangle \, \psi(\mathbf{k}).$$

(i) What is the condition on $\psi(\mathbf{k})$ such that $|\psi\rangle$ be normalised?

(ii) What is the probability that the energy of this particle is less than E?

(iii) What is the state $a(\mathbf{k}) |\psi\rangle$?

(iv) A spacetime-dependent wavefunction associated with $|\psi\rangle$ is defined as

$$\Psi(x) = \langle 0| \, \phi(x) \, |\psi\rangle \, .$$

(a) Show that $\Psi(x)$ satisfies the Klein-Gordon equation.
(b) Express $\Psi(x)$ in terms of $\psi(\mathbf{k})$.
(c) Show that the density

$$\rho(x) = i\Psi^*(x)\partial_0\Psi(x) - i\Psi(x)\partial_0\Psi^*(x)$$

satisfies $\int \mathrm{d}^3x \, \rho(x^0, \vec{x}) = 1$ if $\langle\psi|\psi\rangle = 1$.

Problem 2. (i) Evaluate the matrix elements between the one-particle states

$$\langle \mathbf{p}| : \phi(x)\phi(y) : |\mathbf{q}\rangle \quad \text{and} \quad \langle \mathbf{p}| : \partial_\mu\phi(x)\partial_\nu\phi(x) : |\mathbf{q}\rangle$$

where ϕ is a real free scalar field.

(ii) Calculate the expectation value of

$$T_{\mu\nu} = \, : \partial_\mu\phi\partial_\nu\phi : - \eta_{\mu\nu} : \frac{1}{2}\left(\partial_\rho\phi\partial^\rho\phi - m^2\phi^2\right) :$$

in the state $|\psi\rangle$ of Problem 1 and express it in terms of $\Psi(x)$.

Problem 3. Define

$$(f, g) \equiv \int \frac{đ^3k}{2E_{\mathbf{k}}} f^*(\mathbf{k}) g(\mathbf{k})$$

for any two functions f, g at $\mathbf{k}$. A two-fermion state each of spin 1/2 is defined by

$$|\chi\rangle = \int \frac{đ^3k_1}{2E_{\mathbf{k}_1}} \frac{đ^3k_2}{2E_{\mathbf{k}_2}} |\mathbf{k}_1, \mathbf{k}_2\rangle \, f(\mathbf{k}_1) g(\mathbf{k}_2)$$

where $(f,f) = (g,g) = 1$ and $(f,g) = 0$ (as both fermions are of spin 1/2 the spin labels have been dropped).

(i) Show that $\langle\chi|\chi\rangle = 1$.

(ii) Write down the expression for $\langle\mathbf{k}_1, \mathbf{k}_2|\chi\rangle$.

(iii) Evaluate the expressions

$$\langle\chi|\, a^\dagger(\mathbf{k})\, |\psi\rangle \quad \text{and} \quad \langle\psi|\, a(\mathbf{k})\, |\chi\rangle$$

where $|\psi\rangle = \displaystyle\int \frac{đ^3k}{2E_\mathbf{k}}\, |\mathbf{k}\rangle\, \psi(\mathbf{k})$.

Problem 4. In the Coulomb gauge the only non-vanishing equal-time commutator between components of A_μ and its time-derivatives is

$$[A^i(x), \partial_0 A^j(y)]\Big|_{x^0=y^0} = i\,\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) + i\,\partial^i\partial^j \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$$

(i) Use this formula to evaluate the equal time commutators between components of electric and magnetic fields.

(ii) The covariant commutation relations of A^μ in the Lorenz gauge are

$$[A^\mu(x), A^\nu(y)] = -i\,\eta^{\mu\nu} D(x-y)$$

(a) Evaluate $D(x-y)$

(b) Use this to express the commutator $[F_{\mu\nu}(x), F_{\rho\sigma}(y)]$ in terms of $D(x-y)$.

(iii) Use the results of (ii)(b) together with

$$D(x)\Big|_{x^0=0} = 0 \quad \text{and} \quad \partial_0 D(x)\Big|_{x^0=0} = -\delta^{(3)}(\mathbf{x})$$

to verify the canonical commutation relations of the electric and magnetic fields in (i).

Problem Set 6

Quantum Field Theory

Problem 1. Let A and B be operators such that $[A, B] = c\mathbb{1}$, where $c \in \mathbb{C}$. Show that,

(i) $e^{\lambda A}\, B\, e^{-\lambda A} = B + \lambda c \mathbb{1}$.

(ii) $e^{\lambda A}\, e^{\mu B} = e^{\mu B}\, e^{\lambda A}\, e^{\lambda \mu c}$.

(iii) $e^{\lambda A + \mu B} = e^{\mu B}\, e^{\lambda A}\, e^{\frac{1}{2}\lambda \mu c}$.

Problem 2. A magnetic dipole is described by the vector potential,

$$\underline{A}(\underline{x}) = \underline{\mu} \times \underline{\partial}\, \frac{1}{4\pi|\underline{x}|} = -\frac{\underline{\mu} \times \underline{x}}{4\pi|\underline{x}|^3},$$

where $\underline{\mu}$ is the (constant) dipole moment vector.

(i) Compute the Fourier transform $\underline{A}(\underline{k})$.

(ii) Calculate the scattering matrix element

$$\langle \underline{p}', i' |\, S\, | \underline{p}, i \rangle$$

for elastic scattering of an electron[1] by the dipole field.

(iii) Writing the S-matrix element as,

$$\langle \underline{p}', i' |\, S\, | \underline{p}, i \rangle = \delta_{i'i}\, 2E_p \delta^{(3)}(\underline{p}' - \underline{p}) + \delta(E_p - E_{p'}) i\mathcal{M}_{i'i}(p', p),$$

the differential cross-section of the process is,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{16\pi^2} \sum_{i'} \frac{1}{2} \sum_{i} \left| \mathcal{M}_{i'i}(p', p) \right|^2$$

for scattering that[2] the initial and final spins are <u>not</u> measured. Compute $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ for the case that $\underline{p}$ is parallel to $\underline{\mu}$, i.e. the incident momentum $\underline{p}$ is along the axis of the dipole.

Problem 3.

(i) Give the Feynman rules of the $:\frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4:$ theory in configuration space.

(ii) Give the Feynman diagrams for the connected 3-point function in configuration space up to and including 1-loop corrections.

(iii) Give the analytic expression of the Feynman diagrams in part (ii) including symmetry factors.

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[1]**N.B.**: In the sheet the professor has written what I could only interpret as "elector". At this time I am considering it to be a typo, subject to change as the course progresses.

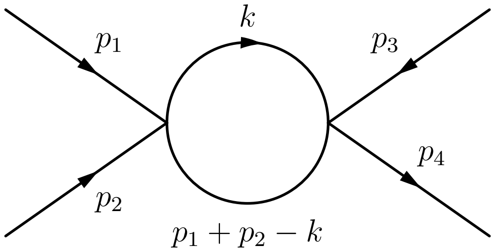
[2]No idea what he means here, could be typo for "where".

Problem Set 7

Quantum Field Theory

Problem 1. Show that in the n-point function of an interacting scalar field theory, only connected diagrams contribute, i.e. the vacuum to vacuum sub-diagrams cancel.

Problem 2. Consider the correction to the 4-point function of ϕ^4 theory described by the diagram[1],



Using a Wick rotation to Euclidean space and the formulae

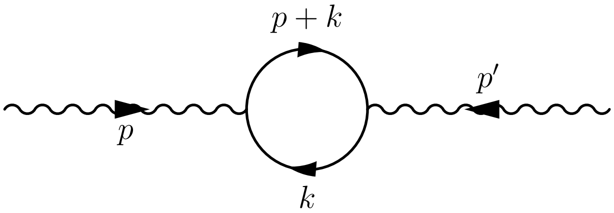
$$\frac{1}{K_E^2 + m^2} = \int_0^\infty \mathrm{d}s \, e^{-s(K_E^2 + m^2)},$$

and,

$$\int \mathrm{d}^D k \; e^{-\frac{1}{2}k^T A k + B^T k} = \frac{(2\pi)^{\frac{D}{2}}}{\sqrt{\det A}} \; e^{\frac{1}{2} B^T A^{-1} B},$$

to integrate the internal momentum, find the divergence of the diagram using dimensional regularization.

Problem 3. The 1-loop correction to the photon's self-energy is,



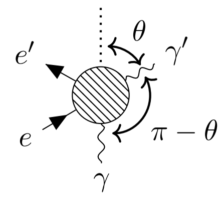
Calculate the diagram and show that it is proportional to $(p_\mu p_\nu - \eta_{\mu\nu} p^2)$.

[1] **N.B.**: I have drawn the arrow on p_2 exactly as given in the sheet, but I think it is a typo and should be the other way around.

Problem Set 8

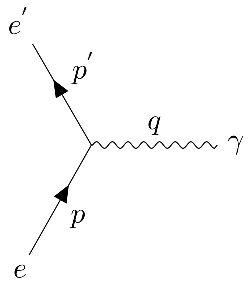
Quantum Field Theory

Problem 1. Consider the scattering process



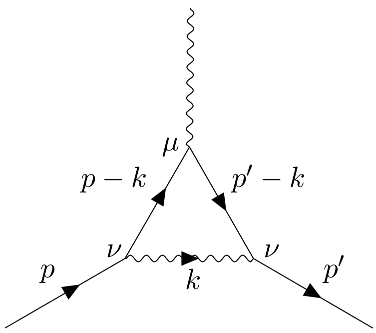
where a photon γ and an electron e scatter to a photon γ' and an electron e'. Use conservation of the 4-momentum to express the wavelength of the photon γ in terms of that of γ' and the scattering angle θ as indicated in the diagram. Perform the computation in the Lorentz frame in which e is at rest.

Remark. The process



where γ is a photon, e is an electron and $q \neq 0$ is not allowed in QED on momentum conservation grounds. Indeed, working in the frame $p = (m, 0, 0, 0)$, conservation of momentum gives $q = p - p'$. Since q is null we have that $q^2 = (p - p')^2 = p^2 - 2pp' + (p')^2 = 0$ which implies $m^2 - 2mE_{p'} + m^2 = 0$ giving $E_{p'} = m$. This is a contradiction as $E_{p'} = \sqrt{\vec{p}^2 + m^2} > m$ if $q \neq 0$.

Problem 2. Consider the following 1-loop correction to the interaction vertex in QED:

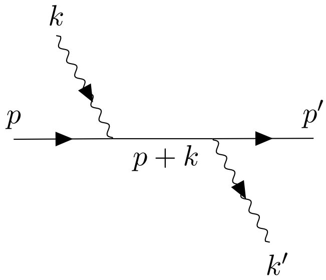


Compute the diagram in dimensional regularization. You may use without proof that

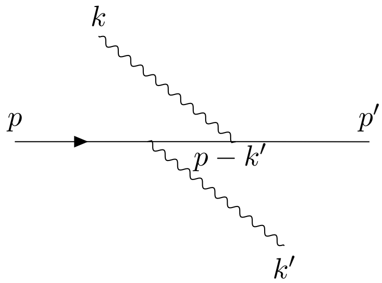
$$
\begin{aligned}
\frac{1}{abc} &= 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}z \frac{1}{[a+(b-c)x+(c-a)z]} \\
\int \mathrm{d}^D k \frac{k^\mu}{(k^2-s+i\varepsilon)^n} &= 0 \\
\int \mathrm{d}^D k \frac{1}{(k^2-s+i\varepsilon)^n} &= i\pi^{D/2}(-1)^n \frac{\Gamma(n-D/2)}{\Gamma(n)} \frac{1}{s^{n-D/2}} \\
\int \mathrm{d}^D k \frac{k^\mu k^\nu}{(k^2-s+i\varepsilon)^n} &= i\pi^{D/2}(-1)^{n+1} \frac{\Gamma(n-D/2-1)}{2\Gamma(n)} \frac{g^{\mu\nu}}{s^{n-D/2-1}}
\end{aligned}
$$

where $g_{\mu\nu}$ is the Minkowski space metric.

Problem 3. At the lowest order, the diagrams that contribute to the Compton scattering are



(a)



(b)

where is a fermion (electron) propagator and is a photon propagator.

(i) Use the Feynman rules to find the S-matrix elements for these processes.

(ii) Writing the S-matrix as $S = \not{\delta}(p+k-p'-k')\, i\,(\mathcal{M}_a + \mathcal{M}_b)$ where $\mathcal{M}_a$ and $\mathcal{M}_b$ contributions correspond to the diagrams (a) and (b), the differential cross section is proportional to

$$A = \frac{1}{4} \sum_{\text{polarization}} \sum_{\text{spins}} |\mathcal{M}|^2$$

where $\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$ and we have averaged over initial and finals spins and polarizations. Show that

$$A = \frac{e^4}{16}\left(\frac{X_{aa}}{(pk)^2} + \frac{X_{bb}}{(pk')^2} - \frac{X_{ab}+X_{ba}}{(pk)(pk')} \right)$$

(impose the on-shell relations for p, p', k and k') where

$$\begin{aligned}
X_{aa} &= \mathrm{tr}\left\{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)\gamma_\mu(\not{p}'+m)\right\} \\
X_{bb} &= \mathrm{tr}\left\{\gamma^\mu(\not{p}-\not{k}'+m)\gamma^\nu(\not{p}+m)\gamma_\nu(\not{p}-\not{k}'+m)\gamma_\mu(\not{p}'+m)\right\} \\
X_{ab} &= \mathrm{tr}\left\{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu(\not{p}+m)\gamma_\mu(\not{p}-\not{k}'+m)\gamma_\nu(\not{p}'+m)\right\} \\
X_{ba} &= \mathrm{tr}\left\{\gamma^\mu(\not{p}-\not{k}'+m)\gamma^\nu(\not{p}+m)\gamma_\mu(\not{p}+\not{k}+m)\gamma_\nu(\not{p}'+m)\right\}.
\end{aligned}$$

(iii) Compute X_{aa}, X_{bb}, X_{ab} and X_{ba} in the frame that $p = (m, 0, 0, 0)$.

(iv) Use your results to find A.